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A new approach to study limit cycles on a cylinder

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ABSTRACT. We present a new approach to study limit cycles of planar systems of autonomous differential equations with a cylindrical phase space Z. It is based on an extension of the Dulac function which we call Dulac-Cherkas function Ψ. The level set $W := \{\varphi,y) \in Z : \Psi(\varphi,y) = 0\}$ plays a key role in this approach, its topological structure influences existence, location and number of limit cycles. We present two procedures to construct Dulac-Cherkas functions. For the general case we describe a numerical approach based on the reduction to a linear programming problem and which is implemented by means of the computer algebra system Mathematica. For the class of generalized Liénard systems we present an analytical approach associated with solving linear differential equations and algebraic equations.

1. INTRODUCTION

We consider systems of two scalar autonomous differential equations

$$\frac{dx}{dt} = P(x,y), \qquad \frac{dy}{dt} = Q(x,y), \tag{1.1}$$

where P and Q are periodic in x with period 2π. Under this assumption we can identify the phase space of (1.1) with the cylinder $Z := S^1 \times R$, where S^1 is the unit circle. Interpreting x as arclength φ on S^1 we will use for the sequel the notation

$$\frac{d\varphi}{dt} = P(\varphi,y), \qquad \frac{dy}{dt} = Q(\varphi,y). \tag{1.2}$$

Let γ_1 and γ_2 be two closed curves on Z which do not intersect and which are not contractible to a point, that is, they surround the cylinder Z. We denote by Ω the finite region on Z bounded by γ_1 and γ_2 .
An isolated periodic solution of (1.2) with some minimal period is called a limit cycle. It is well known that we have to distinguish two kinds of limit cycles of (1.2) in Ω. A limit cycle $\Gamma \in \Omega$ is called a limit cycle of the first kind, if Γ is contractible to a point in Ω, Γ is called a limit cycle of the second kind if Γ surrounds the cylinder Z, that means, it is not contractible to a point in Ω [1, 7].

To investigate limit cycles of the first kind, the well-known methods for planar autonomous systems (1.1) can be applied (see e.g. [2, 8]). Especially, the existence of a limit cycle of the first kind of system (1.2) in Ω requires the existence of an equilibrium of (1.2) in Ω. In contrast to that fact, the existence of a limit cycle of the second kind in Ω does not need the existence of any equilibrium in Ω.
One method to investigate the existence (non-existence) of limit cycles of the first kind and to estimate their number is based on the construction of a Dulac function and its generalization [3]. The method of Dulac function can be also applied to investigate limit cycles of the second kind, see e.g. [2]. In what follows we want to show that also generalizations of the method of Dulac function [3, 6] can be used to study limit cycles of the second kind of system (1.2).

The paper is organized as follows: In section 2 we introduce the notation of a Dulac-Cherkas function Ψ and characterize its relationship to a Dulac function. Section 3 is devoted to some properties of the curve $W := \{(\varphi,y) \in \Omega : \Psi(\varphi,y) = 0\}$. In section 4 we exploit the topological structure of the branches of the curve W to derive results on the existence of limit cycles of the second kind, to estimate their number and location and to characterize their hyperbolicity and stability. Section 5 deals with the numerical construction of a Dulac-Cherkas function for system (1.2) in form of a linear combination of some base functions by using the reduction to a linear programming problem. In section 6 we present an algorithm to construct a class of Dulac-Cherkas functions for the second order differential equation

$$\frac{d^2\varphi}{dt^2} = \sum_{j=0}^{l} h_j(\varphi)\Big(\frac{d\varphi}{dt}\Big)^j \tag{1.3}$$

by solving linear differential equations and algebraic equations.

2. Assumptions, Definitions, Preliminaries

Throughout the sections 2 - 5 we assume

(A_1). P and Q belong to the class $C^1(\Omega,R)$, and are 2π-periodic in the first variable.

Let f be the vector field defined on Ω by system (1.2), let D be a subregion of Ω.

Definition 2.1. *A function $B\in C^1(D,R)$ with the properties*

(i). $B(\varphi,y) = B(\varphi+2\pi,y) \quad \forall(\varphi,y)\in D,$
(ii). $div(Bf) := \frac{\partial(BP)}{\partial\varphi} + \frac{\partial(BQ)}{\partial y} = (gradB, f) + Bdivf \geq 0 \ \ (\leq 0) \ \ in \ \ D,$
where $div(Bf)$ vanishes only on a subset of D of measure zero

is called a Dulac function of system (1.2) in D.

The following result can be found in [2].

Theorem 2.2. *Let B be a Dulac function of (1.2) in D. If the boundary ∂D of D is connected and contractible to a point, then (1.2) has no limit cycle in D. In case that ∂D consists of two closed curves in Ω which do not intersect and which are not contractible to a point, then (1.2) has no limit cycle of the first kind in D and at most one limit cycle of the second kind of (1.2) in D.*

Now we introduce a generalization of a Dulac function which we call Dulac-Cherkas function.

Definition 2.3. *A function $\Psi\in C^1(D,R)$ with the properties*

(i). $\Psi(\varphi,y) = \Psi(\varphi+2\pi,y) \quad \forall(\varphi,y)\in D,$
(ii). *The set $W := \{(\varphi,y)\in D : \Psi(\varphi,y) = 0\}$ has measure zero,*
(iii). *There is a real number $k\neq 0$ such that*

$$\Phi(\varphi,y) := (grad\Psi, f) + k\Psi divf \geq 0 \ \ (\leq 0) \quad in\, D, \tag{2.1}$$

where the set $V := \{(\varphi,y)\in D : \Phi(\varphi,y) = 0\}$ has the properties
(a). *V has measure zero,*
(b). *If Γ is a limit cycle of (1.2), then it holds $\Gamma\cap V\neq\Gamma$,*

(iv).

$$(grad\Psi, f)_{|W} \neq 0 \tag{2.2}$$

is called a Dulac-Cherkas function of (1.2) in D,

Remark 2.4. *If the inequalities in (2.1) hold strictly, then also condition (2.2) is valid.*

Remark 2.5. *If Φ does not depend on y, that is $\Phi(\varphi, y) \equiv \Phi_0(\varphi)$ and if Φ_0 vanishes only in finitely many points φ_i in $[0, 2\pi]$, then the conditions on the set V are fulfilled.*

In order to give an example of a Dulac-Cherkas function, we consider the system

$$\frac{d\varphi}{dt} = y, \quad \frac{dy}{dt} = \sin\varphi + 3\cos\varphi - \frac{61}{2} + \Big(3\cos\varphi - \frac{57}{2}\Big)y + 3y^2 + y^3 \tag{2.3}$$

and introduce the function $\Psi : Z \to R$ by

$$\Psi(\varphi, y) := \frac{y^2}{2} + y + \cos\varphi - 10. \tag{2.4}$$

Setting $k = -2/3$ we get from (2.1)

$$\Phi(\varphi, y) \equiv \Phi_0(\varphi) = -\frac{441}{2} + 42\cos\varphi + \sin\varphi - 2\cos^2\varphi < 0 \quad \text{for} \quad 0 \leq \varphi \leq 2\pi. \tag{2.5}$$

By Remark 2.4 and Remark 2.5 all conditions in Definition 2.3 are fulfilled. Therefore, Ψ is a Dulac-Cherkas function of (2.3) in Z.

Lemma 2.6. *From (2.1) and (2.2) we get*

$$\Phi(\varphi, y)_{|W} = (grad\Psi, f)_{|W} > 0 \quad (< 0), \tag{2.6}$$

that is, $d\Psi/dt$ has on all branches of W the same sign as the function Φ.

A relationship between a Dulac function and a Dulac-Cherkas function is described in the following lemma.

Lemma 2.7. *Let Ψ be a Dulac-Cherkas function of (1.2) in D. Let $\tilde{D}$ be a subregion of D, where Ψ is either positive or negative. Then $B := |\Psi|^{\frac{1}{k}}$ is a Dulac function in $\tilde{D}$.*

Proof. By (2.1) we have

$$div(Bf) = div\Big(|\Psi|^{\frac{1}{k}} f\Big) = \frac{1}{k}|\Psi|^{\frac{1}{k}-1} sign\Psi\Big[(grad\Psi, f) + k\Psi div f\Big]$$

$$= sign\Psi \frac{1}{k}|\Psi|^{\frac{1}{k}-1}\Phi \geq 0 \quad (\leq 0).$$

Since Φ vanishes only on a set of measure zero, the proof is complete. □

In the next section we derive some properties of a Dulac-Cherkas function which we exploit to estimate the number of limit cycles of the second kind.

3. Properties of a Dulac-Cherkas function

Let Ψ be a Dulac-Cherkas function of system (1.2) in D. In case that the set W is empty, that is we have $\Psi > 0$ ($\Psi < 0$) in D, the function $|\Psi|^{\frac{1}{k}}$ represents by Lemma 2.7 a Dulac function and we can restrict ourselves to methods based on a Dulac function. Therefore, in what follows we assume that the set W is not empty. Our goal is to investigate some properties of the curve W.
First we prove the following transversality result.

Lemma 3.1. *Any trajectory of system (1.2) meeting the curve W intersects W transversally.*

Proof. We denote by $\frac{d\Psi}{dt}$ the derivative of the function Ψ along system (1.2). From (2.2) we get

$$\frac{d\Psi}{dt}_{|W} = (grad\Psi, f)_{|W} \neq 0.$$

□

Corollary 3.1. *The function Ψ changes its sign when crossing W.*

Lemma 3.2. *The curve W does not contain any equilibrium of system (1.2).*

Proof. Let E be an equilibrium of (1.2), that is $f(E) = 0$. Suppose $E \in W$, then we get from (2.1) $\Phi(E)_{|W} = 0$ which contradicts to (2.6). □

Lemma 3.3. *Let W_1 and W_2 be two smooth local open branches of the curve W such that $\partial W_1 \cap \partial W_2$ is empty. Then W_1 and W_2 do not meet.*

Proof. Suppose W_1 and W_2 meet at the point M. Then M is an interior point with respect to W_1 and with respect to W_2. Let γ_M be the trajectory of (1.2) passing through $M \in W$. According to Lemma 3.1, γ_M intersects W_1 and W_2 transversally in M. Since $\Phi(M) \neq 0$ by (2.6), we may assume for definiteness $\Phi(M) = (grad\Psi(M), f(M)) > 0$. Thus, there is a small neighborhood $\mathcal{N}_M$ of M in D with the following properties:
(i) $(grad\Psi, f) > 0$ in $\mathcal{N}_M$.
(ii) $W_1 \cap \mathcal{N}_M$ and $W_2 \cap \mathcal{N}_M$ are connected sets.
(iii) $(W_1 \cap \mathcal{N}_M) \cap (W_2 \cap \mathcal{N}_M) = M$.
(iv). All trajectories of (1.2) which are sufficiently near to γ_M intersect W_1 and W_2 transversally in $\mathcal{N}_M$.
Let γ be such a trajectory intersecting W_1 transversally at the point M_1 in $\mathcal{N}_M$ and W_2 transversally at the point M_2 in $\mathcal{N}_M$. Let g be the solution of (1.2) representing γ such that $g(t_1) = M_1$ and $g(t_2) = M_2$, where we assume $t_2 > t_1$ and $g(t) \in \mathcal{N}_M$ for $t_1 \leq t \leq t_2$. If we introduce the function $h(t) := \Psi(g(t))$, then we have

$$h(t_1) = \Psi(M_1) = h(t_2) = \Psi(M_2) = 0.$$

Concerning the derivative of h along system (1.2) we have

$$\frac{dh}{dt}(t) = (grad\Psi(g(t)), f(g(t))) > 0 \quad \text{for} \quad t_1 \leq t \leq t_2.$$

Thus, from the relation $h(t_1) = 0$ we get $h(t_2) > 0$ for $t_2 > t_1$. The obtained contradiction implies that W_1 and W_2 do not meet. □

Corollary 3.1 and Lemma 3.3 imply immediately the result:

Lemma 3.4. *The curve W decomposes the region D in subregions on which Ψ is definite and where the transition from one subregion to an adjacent subregion is connected with a sign change of Ψ.*

From lemma 3.4 we get the following result:

Theorem 3.5. *Let Ψ be a Dulac-Cherkas function of system* (1.2) *in $D \subset \Omega$. Then any limit cycle of system* (1.2) *which is entirely located in D does not intersect the curve W.*

Proof. Let Γ be a limit cycle of system (1.2) entirely located in D. We assume that Γ intersects W. Since W decomposes D into subregions where Ψ is definite and since Γ is a closed curve, Γ must intersect W twice. But this is impossible by Lemma 2.6.

□

According to Theorem 3.5, the topological structure of the branches of the curve W strongly influences the localization of the limit cycles of system (1.2) completely located in D. This will be studied in the next section.

4. On the limit cycles of the second kind

For the sequel we assume that the boundary of the region $D \subset \Omega$ is formed by the curves Δ_1 and Δ_2 surrounding the cylinder Z, and that Δ_1 and Δ_2 have the representations $\Delta_1 := \{(\varphi, y) \in \Omega : y = \delta_1(\varphi), 0 \leq \varphi \leq 2\pi\}$ and $\Delta_2 := \{(\varphi, y) \in \Omega : y = \delta_2(\varphi), 0 \leq \varphi \leq 2\pi\}$, respectively, where δ_1 and δ_2 are 2π-periodic functions. Without loss of generality we suppose $\delta_2(\varphi) < \delta_1(\varphi)$ for all φ.

Theorem 4.1. *Let Ψ be a Dulac-Cherkas function of (1.2) in D. Then it holds:*
(i). If the set W is empty, then system (1.2) has at most on limit cycle of the second kind in D.
(ii). If the set W contains at least two branches connecting the curves Δ_1 and Δ_2, then system (1.2) has no limit cycle of the second kind in D.
(iii) If the curve W consists in D of s closed branches (ovals) $W_1, W_2, ..., W_s$ surrounding the cylinder Z and if D contains no equilibrium of (1.2), then system (1.2) has at least $s-1$ but not more than $s+1$ limit cycle of the second kind in D.

Proof. If W is empty, then $|\Psi|^{\frac{1}{k}}$ is a Dulac function in D. Hence, the claim (i) follows from Theorem 2.2.
Now we suppose that W contains at least two branches W_1 and W_2 connecting the curves Δ_1 and Δ_2. By Theorem 3.5, any limit cycle of (1.2) does neither meet W_1 nor W_2. Thus, there is no limit cycle of the second kind of (1.2) in D.
To prove (iii) we assume that W consists of s ovals $W_1, ..., W_s$, where the oval W_i, $1 \leq i \leq s$, has the representation $W_i := \{(\varphi, y) \in D : y = w_i(\varphi), 0 \leq \varphi \leq 2\pi\}$. Without loss of generality we may assume

$$\delta_1(\varphi) > w_1(\varphi) > w_2(\varphi) > ... > w_s(\varphi) > \delta_2(\varphi) \quad \text{for} \quad 0 \leq \varphi \leq 2\pi.$$

For $1 \le i \le s-1$ we denote by D_i the (open) strip bounded by W_i and W_{i+1}, the strip bounded by Δ_1 and W_1 is denoted by D_0, the strip bounded by W_s and Δ_2 is denoted by D_s. By Lemma 3.4, the function Ψ is different from zero in any strip D_j, $0 \le j \le s$, and changes its sign when crossing any oval W_j. According to Lemma 2.7, $|\Psi|^{\frac{1}{k}}$ is a Dulac function in $D_j, 0 \le j \le s$. Therefore, by Theorem 2.2 any strip D_j contains at most one limit cycle of the second kind. For definiteness we assume $\Phi \ge 0$ in D and $\Psi < 0$ in D_0. Then $(grad\Psi, f)$ is strictly positive on all ovals $W_1, ..., W_s$. Thus, any trajectory of (1.2) crossing the boundary of each of the strips $D_1, D_3, ...$ enters it for increasing t, and any trajectory of (1.2) crossing the boundary of each of the strips $D_2, D_4, ...$ leaves it for increasing t. Therefore, the strips $D_1, D_3, ...$ contain a unique limit cycle of the second kind which is asymptotically orbitally stable, and the strips $D_2, D_4, ...$ contain a unique limit cycle of the second kind which is orbitally unstable. Moreover, the strips D_0 and D_s might contain at most one limit cycle of the second kind. This completes the proof.

□

Theorem 4.1 contains no assumption on the sign of k. The following theorem shows that the topological structure of the branches of the curve W described in case (iii) of Theorem 4.1 is only possible for negative k.

Theorem 4.2. *Let Ψ be a Dulac-Cherkas function of (1.2) in D such that the set W contains $s \ge 2$ ovals $W_1, ..., W_s$ surrounding the cylinder. Then it holds*
(i). The number k in the expression (2.1) for Φ is negative.
(ii). The unique limit cycle of the second kind located in the strip D_j bounded by two consecutive ovals W_j and W_{j+1} is hyperbolic.

Proof. For definiteness we suppose $\Phi \ge 0$ in D and $\Psi > 0$ in D_j. From the proof of Theorem 4.1 we get that the strip D_j contains a unique limit cycle Γ of the second kind which is asymptotically orbitally stable. Thus it holds

$$\oint_\Gamma divf ds \le 0, \tag{4.1}$$

where the equality occurs only in the case that Γ is a multiple limit cycle.

From (2.1) we get any trajectory of (1.2) crossing the boundary

$$divf = \frac{\Phi - \frac{d\Psi}{dt}}{k\Psi}$$

such that we have

$$\oint_\Gamma divf ds = \frac{1}{k}\oint_\Gamma \frac{\Phi}{\Psi} ds.$$

By $\Phi \ge 0$ in D and since Ψ is positive in D_j and $\Gamma \cap V \ne \Gamma$ by Definition 2.3, we have

$$\oint_\Gamma \frac{\Phi}{\Psi} ds > 0.$$

Hence, it holds

$$\oint_\Gamma divf ds < 0,$$

$$sign \oint_{\Gamma} divf ds = sign k.$$

Hence, the claims (i) and (ii) hold true.

□

In the following section we present a numerical approach to construct a Dulac-Cherkas function for the general system (1.2) in a closed finite region $\Omega \subset Z$.

5. COMPUTER ASSISTED CONSTRUCTION OF A DULAC-CHERKAS FUNCTIONS

Analogously to the construction of a Dulac-Cherkas function for planar systems [3, 5] in some closed finite region Ω we use the ansatz

$$\Psi(\varphi, y) = \sum_{j=1}^{N} c_j \psi_j(\varphi, y), \tag{5.1}$$

where the functions $\psi_j, j = 1, ..., N$, are base functions belonging to the space $C^1(\Omega, R)$, and which are 2π-periodic in φ, $c = (c_1, ..., c_N) \in R^N$ is a vector to be determined. If we plug the ansatz (5.1) into (2.1), then Φ can be also represented in the form

$$\Phi(\varphi, y, c, k) = \sum_{j=1}^{N} c_j \phi_j(\varphi, y, k), \tag{5.2}$$

where the functions $\phi_1, ..., \phi_N$ are some functions determined by the base functions $\psi_1, ..., \psi_N$, their derivatives with respect to φ and y, and the parameter k. If there is a parameter $k \neq 0$ such that it holds

$$\max_{|c|=1} \min_{(\varphi, y) \in \Omega} \Phi(\varphi, y, c, k) > 0, \tag{5.3}$$

where $|.|$ is some norm in R^n, then we can conclude that the function Ψ defined in (5.1) is a Dulac-Cherkas function (5.1) for system (1.2) in Ω. In this way, the construction of a Dulac-Cherkas function can be reduced to the linear programming problem

$$L \to \max, \quad \sum_{j=1}^{N} c_j \phi_j(\varphi_p, y_p) - L \geq 0, \ |c| \leq 1 \tag{5.4}$$

on a grid of nodes $(\varphi_p, y_p), p = 1, \ldots, N_0$, in the region Ω.
As base functions on the cylinder Z we use the functions $y^i cos(l\varphi)$ and $y^i sin(l\varphi)$, where i and l are positive integer, such that we have

$$\Psi(\varphi, y) := \sum_{i=1}^{n+1} y^{i-1} \Big[\sum_{l=1}^{m} \big(a_{il} cos((l-1)\varphi) + b_{il} sin((l-1)\varphi) \big) \Big]. \tag{5.5}$$

As an example we consider the system

$$\frac{d\varphi}{dt} = y^2 + 1, \quad \frac{dy}{dt} = (y - 1.5 - 0.3 sin\varphi)(y + 1.5 + 0.3 sin\varphi)(y - 0.3 sin\varphi), \tag{5.6}$$

which has no equilibrium and therefore no limit cycle of the first kind. Our goal is to estimate the number of limit cycles of the second kind for system (5.6) in the region $\Omega : -\pi \leq \varphi \leq \pi, \quad -2\pi \leq y \leq 2\pi$. For this purpose we use the ansatz (5.5) with

$m = 4, n = 4$ and set $k = -0.5$ in the corresponding function Φ. We cover Ω with a uniform grid containing 900 nodes. By solving the linear programming problem (5.4) we get 20 constants a_{il} and 20 constants b_{il}. Using these constants, we get from (5.5) and (1.2) that $\Phi(\varphi, y, a_{il}, b_{il}, -0.5) > 0.4$ is strictly positive in Ω, and the set W consists of 4 ovals W_1, W_2, W_3, W_4 surrounding the cylinder and which define the strips D_0, D_1, D_2, D_3, D_4 (see Fig. 1). The light strips D_0, D_2, D_4 correspond to $\Psi > 0$, the black strips D_1 and D_3 correspond to $\Psi < 0$. According to Theorem 4.1, each of the strips D_1, D_2, D_3 contains exactly one limit cycle of the second kind of system (5.6). That is, our numerical study shows that system (5.6) has at least three limit cycles in Ω.

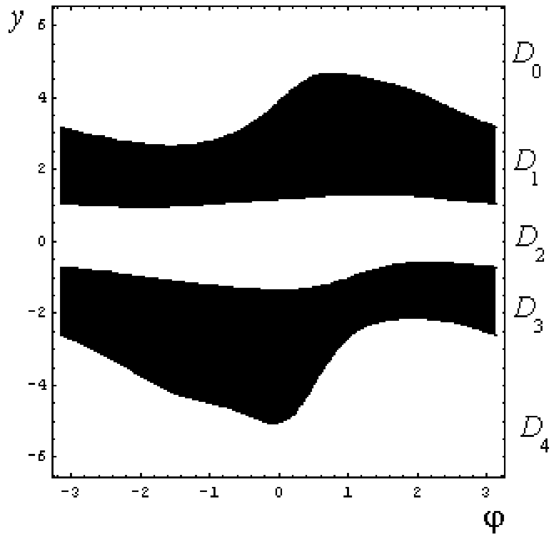

Fig. 1

The following section is devoted to the problem of analytical construction of a Dulac-Cherkas function for a class of autonomous systems with cylindrical phase space which originates from the pendulum equation [2].

6. Algorithm to construct a function Ψ such that Φ does not depend on y

We consider the generalized Liénard system

$$\frac{d\varphi}{dt} = y, \quad \frac{dy}{dt} = \sum_{j=0}^{l} h_j(\varphi) y^j, \quad l \geq 1 \tag{6.1}$$

on the cylinder Z under the assumption
(A_2). The functions $h_0, ..., h_l$ are continuous on R and 2π-periodic, where

$$h_l(\varphi) \not\equiv 0. \tag{6.2}$$

The corresponding vector field will be denoted by f_l.
It can be easily verified that under the condition (A_2), which is weaker than condition (A_1), a Dulac-Cherkas function Ψ for system (6.1) can be defined on the whole

cylinder Z. For Ψ we make the ansatz

(6.3) $$\Psi(\varphi,y)=\sum_{j=0}^{n}\Psi_j(\varphi)y^j, \quad n\geq 1$$

with

(6.4) $$\Psi_n(\varphi)\not\equiv 0,$$

where we assume that the functions $\Psi_0,...,\Psi_n$ are continuously differentiable and 2π-periodic.

Analogously to the construction of a Dulac-Cherkas function for generalized Liénard system in the plane [4] we describe an algorithm to determine the functions Ψ_j in (6.3) and the constant k such that the corresponding function Φ determined by

(6.5) $$\Phi(\varphi,y):=(grad\,\Psi(\varphi,y),f_l(\varphi,y))+k\Psi(\varphi,y)\,div\,f_l(\varphi,y)$$

does not depend on y, that is, $\Phi(\varphi,y)\equiv\Phi_0(\varphi)$. By this way, the inequalities in (2.1) are valid globally in y, and we have the possibility to get global estimates on the number of limit cycles.
If we put (6.3) into the right hand side of (6.5) and take into account that the vector field f_l is determined by (6.1) we get

(6.6) $$\begin{aligned}\Phi(\varphi,y)\equiv&\Big(\Psi_0'(\varphi)+\Psi_1'(\varphi)y+...+\Psi_n'(\varphi)y^n\Big)y\\&+\Big(\Psi_1(\varphi)+2\Psi_2(\varphi)y+...+n\Psi_n(\varphi)y^{n-1}\Big)\\&\times\Big(h_0(\varphi)+h_1(\varphi)y+...+h_l(\varphi)y^l\Big)\\&+k\Big(\Psi_0(\varphi)+\Psi_1(\varphi)y+...+\Psi_n(\varphi)y^n\Big)\\&\times\Big(h_1(\varphi)+2h_2(\varphi)y+...+lh_l(\varphi)y^{l-1}\Big).\end{aligned}$$

For the sequel we represent $\Phi(\varphi,y)$ in the form

(6.7) $$\Phi(\varphi,y)\equiv\sum_{i=0}^{m}\Phi_i(\varphi)y^i,$$

where $\Phi_0,...,\Phi_m$ are 2π-periodic continuous functions depending on $h_0,...,h_l$, $\Psi_0,...,\Psi_n$, $\Psi_0',...,\Psi_n'$, and of k.
Concerning the highest power m of y in (6.7) we get from (6.6)

(6.8) $$m=max\{n+1,n+l-1\}.$$

Our goal is to determine the functions $\Psi_j, j=0,...,n$, and the real number k in such a way that we have

(6.9) $$\Phi_i(\varphi)\equiv 0 \quad \text{for} \quad i=1,...,m.$$

Then it holds

(6.10) $$\Phi(\varphi,y)\equiv\Phi_0(\varphi):=\Psi_1(\varphi)h_0(\varphi)+k\Psi_0(\varphi)h_1(\varphi).$$

If we additionally require

(6.11) $$\Phi_0(\varphi) \geq 0 \quad (\leq 0) \quad \text{for} \quad 0 \leq \varphi \leq 2\pi,$$

where Φ_0 vanishes only at finitely many points in $[0, 2\pi]$, and if the inequality (2.2) is valid, then Ψ is a Dulac-Cherkas function of (6.1) in Z.
From (6.6)–(6.8) we get that for $l = 1$ and $l = 2$ the relations (6.9) represent a system of $n+1$ linear differential equations to determine the $n+1$ functions $\Psi_j, j = 0, ..., n$. In case $l = 1$, system (6.9) can be solved successively by simple quadratures, starting with Ψ_n. In case $l = 2$, this system can also be integrated by solving inhomogeneous linear differential equations, starting with Ψ_n. The general solution depends on $n+1$ integration constants and on the constant k, but we get no restriction on k in the process of solving this system. An appropriate choice of these constants can imply that Ψ is a Dulac-Cherkas function for (6.1) in Z.

For $l \geq 3$, system (6.9) consists of $l-2$ algebraic equations and $n+1$ linear differential equations

(6.12)
$$\begin{aligned}
0 &\equiv (n+lk)h_l(\varphi)\Psi_n(\varphi),\\
0 &\equiv (k(l-1)+n)h_{l-1}(\varphi)\Psi_n(\varphi) + (n-1+lk)h_l(\varphi)\Psi_{n-1}(\varphi),\\
0 &\equiv (n-1+k(l-1))h_{l-1}(\varphi)\Psi_{n-1}(\varphi)\\
&\quad + (n+k(l-2))h_{l-2}(\varphi)\Psi_n(\varphi) + (n-2+lk)h_l(\varphi)\Psi_{n-2},\\
&\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\\
0 &\equiv \Psi_n'(\varphi) + nh_2(\varphi)\Psi_n(\varphi) + (n-1)h_3(\varphi)\Psi_{n-1}(\varphi) + \ldots\\
&\quad + (n-l)h_{l+2}(\varphi)\Psi_{n-l}(\varphi) + 2kh_2(\varphi)\Psi_n(\varphi)\\
&\quad + 3kh_3(\varphi)\Psi_{n-1}(\varphi) + \ldots + klh_l(\varphi)\Psi_{n-l+2}(\varphi),\\
&\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\\
0 &\equiv \Psi_1'(\varphi) + (1+2k)h_2(\varphi)\Psi_1(\varphi) + 3kh_3(\varphi)\Psi_0(\varphi)\\
&\quad + (2+k)h_1(\varphi)\Psi_2(\varphi) + 3h_0(\varphi)\Psi_3(\varphi),\\
0 &\equiv \Psi_0'(\varphi) + 2kh_2(\varphi)\Psi_0(\varphi)\\
&\quad + (k+1)h_1(\varphi)\Psi_1(\varphi) + 2h_0(\varphi)\Psi_2(\varphi)
\end{aligned}$$

to determine k and the functions $\Psi_0, ..., \Psi_n$. Taking into account (6.2) and (6.4) we get from the first equation in (6.12)

$$k = -\frac{n}{l}.$$

For $l \geq 4$, system (6.12) has generically no solution. In this case we have to derive additional conditions on the functions h_i in order that system (6.12) has a solution such that Φ_0 satisfies (6.11).

Now we apply this algorithm in the case $l = 3$ and $n = 2$ using system (2.3). Here, we have $m = 4$, $k = -2/3$, and system (6.12) without the first equation takes the following form

$$
\begin{aligned}
&0 \equiv \Psi_2'(\varphi) + \frac{2}{3}h_2(\varphi)\Psi_2(\varphi) - h_3(\varphi)\Psi_1(\varphi),\\
(6.13)\qquad &0 \equiv \Psi_1'(\varphi) - \frac{1}{3}h_2(\varphi)\Psi_1(\varphi) - 2h_3(\varphi)\Psi_0(\varphi) + \frac{4}{3}h_1(\varphi)\Psi_2(\varphi),\\
&0 \equiv \Psi_0'(\varphi) - \frac{4}{3}h_2(\varphi)\Psi_0(\varphi) + \frac{1}{3}h_1(\varphi)\Psi_1(\varphi) + 2h_0(\varphi)\Psi_2(\varphi).
\end{aligned}
$$

It is easy to verify that

$$
(6.14)\qquad \Psi_2(\varphi) \equiv \frac{1}{2},\ \Psi_1(\varphi) \equiv 1, \Psi_0(\varphi) \equiv \cos\varphi - 10
$$

is a solution of (6.13) and represents the Dulac-Cherkas function introduced in (2.4) for system (2.3).

In the case of system (2.3), Theorem 4.1 can be used to get the following estimation of all limit cycles on the whole cylinder.

Theorem 6.1. *System (2.3) has on the cylinder Z exactly three limit cycles of the second kind which are hyperbolic.*

Proof. The equation

$$
\Psi(\varphi, y) \equiv \frac{1}{2}y^2 + y + \cos\varphi - 10 = 0
$$

describes two ovals

$$
W_1 := \{(\varphi, y) \in Z : y = -1 + \sqrt{21 - 2\cos\varphi}\}
$$

and

$$
W_2 := \{(\varphi, y) \in Z : y = -1 - \sqrt{21 - 2\cos\varphi}\}
$$

surrounding the cylinder. The corresponding function Φ takes by (2.5) only negative values.

Using Theorem 4.1 we get that the strip D_1 between W_1 and W_2 contains a unique asymptotically orbitally stable limit cycle of second kind. It follows from system (2.3) that there is a positive number y_0 such that

$$
\frac{dy}{dt} > 0 \quad \text{for} \quad y \geq y_0, \qquad \frac{dy}{dt} < 0 \quad \text{for} \quad y \leq -y_0.
$$

Therefore, system (2.3) has no limit cycle in the regions $y > y_0$ and $y < -y_0$. Now we introduce the notation

$$
D_0 := \{(\varphi, y) \in Z : -1 + \sqrt{21 - 2\cos\varphi} < y < y_0\}
$$

and

$$
D_2 := \{(\varphi, y) \in Z : -y_0 < y < -1 - \sqrt{21 - 2\cos\varphi}\}.
$$

In these regions the Dulac-Cherkas function Ψ takes only positive values, that is, it represents a Dulac function there. Thus, in D_0 and in D_2 there exists at most one limit cycle of the second kind, respectively. It can be verified that any trajectory of (2.3), which meets the boundary of D_0 (D_2) leaves D_0 (D_2) for increasing t. Consequently, the strips D_0 and D_2 contain a unique limit cycle, respectively, which is orbitally unstable. Therefore, the statement of the theorem holds. □

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REFERENCES

[1] A.A. ANDRONOV, A.A. VITT, S.E. KHAJKIN, *The theory of oscillations*, Nauka, Moscow, 1981.
[2] N.N. BAUTIN, E.A. LEONTOVICH, *Methods and rules for the qualitative study of dynamical systems on the plane*, Nauka, Moscow, 1990.
[3] L.A. CHERKAS, *Dulac function for polynomial autonomous systems on a plane*, Differential Equations **33** (1997), 692-701.
[4] L.A. CHERKAS, A.A. GRIN, K.R. SCHNEIDER, *On the construction of Dulac-Cherkas functions for generalized Liénard systems*, submitted for publication.
[5] L.A. CHERKAS, A.A. GRIN, *Bendixon-Dulac criterion and reduction to global uniqueness in the problem of estimating the number of limit cycles*, Differential Equations **46** (2010), 61-69.
[6] A.A. GRIN, K.R. SCHNEIDER, *On some classes of limit cycles of planar dynamical systems*, Dynamics of Continuous, Discrete and Impulsive Systems, ser. A, **14** (2007), 641 - 656.
[7] G.A. LEONOV, I.M. BURKIN, A.I. SHEPELYAVYI, *Frequency methods in oscillation theory*, Mathematics and its Applications 357, Kluwer Academic Publishers, Dordrecht, 1996.
[8] L. PERKO, *Differential Equations and Dynamical Systems*, New York, Springer, Third Edition, 2001.